Quantum Logic — 2017-18

Module: Categorical Logics for Quantum Informatics

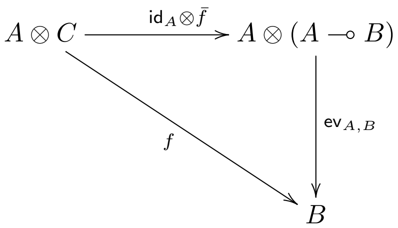
Assessment

Introduction

Refer to [2][1], section 1.7, for the definitions relative to Linear Logic. In particular, see Definition 111 for the rules of $(\otimes, \multimap)$-linear logic, to Definition 113 for the linear λ-calculus typing rules (for function application, use the alternative simpler rule given at the end of that page, on the RHS), and to Section 1.7.3 (especially Table 1.12) for its categorical interpretation in symmetric closed monoidal categories (SCMC). Note: the notation for the *curried* version of f in this reference is $\Lambda(f)$ instead of $\bar{f}$.

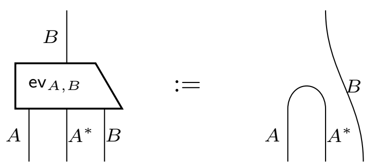
Refer to e.g. [1][2] for the definition of compact closed categories (a subclass of SCMCs): Definition 3 in the linked text. Remember that in diagrammatic notation the η_A correspond to *cups* and the ϵ_A to *caps*.

Remember also that compact closed categories are a subclass of SCMCs, and so in particular they are monoidal closed categories: given objects A, B, there is an object $A \multimap B$ that acts as an internal hom, in that there is an evaluation arrow $\mathsf{ev}_{A,B} : A \otimes (A \multimap B) \longrightarrow B$ with the property that for any arrow $f : A \otimes C \longrightarrow B$, there is an arrow $\bar{f} : C \longrightarrow A \multimap B$ such that the following diagram commutes:

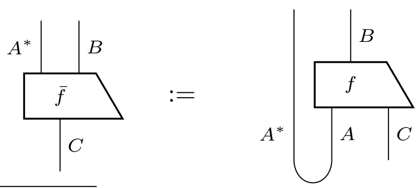


(Note: beware that I've changed the usual ordering here to avoid having to use swap maps above; no essential difference ensues.)

As discussed in the lectures, in a compact closed category, this structure is given by $A \multimap B := A^* \otimes B$, the evaluation map



and, for any arrow $f : A \otimes C \longrightarrow B$, the *curried* arrow $\bar{f} : C \longrightarrow A \multimap B$ given by:



[1] https://arxiv.org/pdf/1102.1313.pdf.
[2] https://arxiv.org/pdf/0808.1023.pdf.

Exercises

1.

For each of the following sequents, either write a derivation tree in Linear Logic or prove that they are not derivable.

(Hint 1: Remember the resource interpretation!)

(Hint 2: To prove that a sequent is not derivable, remember that Linear Logic has Cut Elimination, meaning that if a sequent is derivable then it can be derived without any application of the Cut Rule).

(a) $\vdash A \multimap A$

(b) $A \multimap (B \multimap C) \vdash (A \otimes B) \multimap C$

(c) $\vdash A \multimap (B \multimap A)$

(d) $\vdash A \multimap A \otimes A$

2.

In any compact closed category, given an arrow $f : A \longrightarrow B$, one can define an arrow $f^T : B^* \longrightarrow A^*$ by

$$B^* \;=\; I \otimes B^* \xrightarrow{\eta_A \otimes \mathsf{id}_{B^*}} A^* \otimes A \otimes B^* \xrightarrow{\mathsf{id}_{A^*} \otimes f \otimes \mathsf{id}_{B^*}} A^* \otimes B \otimes B^* \xrightarrow{\mathsf{id}_{A^*} \otimes \epsilon_B} A^* \otimes I \;=\; A$$

Recall the definitions of cups and caps in Rel (category of sets and relations) and $\mathsf{FdVect}_{\mathbb{C}}$ (category of finite-dimentional complex vector spaces and linear maps); for this, see the Examples in between Definition 3 and Definition 4 in [1].

Instantiate the definition of f^T above in each of these two categories:

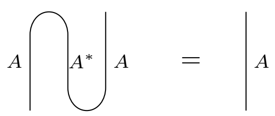
(a) Show that in Rel, given a relation $R : A \longrightarrow B$, the corresponding $R^T : B^* \longrightarrow A^*$ (which has type $R^T : B \longrightarrow A$ because $A^* = A$ for all objects A) is the converse relation of R, i.e.

$$b\, R^T\, a \quad \text{if and only if} \quad a\, R\, b\ .$$

(b) Show that in $\mathsf{FdVect}_{\mathbb{C}}$, given a linear map $f : V \longrightarrow W$, the linear map $f^T : W^* \longrightarrow V^*$ is the transpose of f[3]. In particular, if one chooses particular bases for the vector spaces V and W and f is written as a matrix M, then f^T will be written as the matrix M^T with respect to the corresponding bases for the dual vector spaces V^* and W^*.

Now, working in a general compact closed category:

(Hint: For these, you can use diagrammatic reasoning, in particular the yanking law:



)

[3]see https://en.wikipedia.org/wiki/Transpose_of_a_linear_map.

(c) Show that for any f, we have $(f^T)^T = f$.

(c) Show that the association $f \longmapsto f^T$ is contravariantly functorial. That is, for any object A, $(\mathsf{id}_A)^T = \mathsf{id}_{A^*}$, and for any arrows $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we have $(g \cdot f)^T = f^T \cdot g^T$ (Note: both are arrows of type $C^* \longrightarrow A^*$).

3.

Assuming that the sequent $\Gamma \vdash A$ is derivable, we will consider two derivations of the sequent $\Gamma \vdash B \multimap (B \otimes A)$.

(a) Let u be the term of the linear λ-calculus corresponding to $\Gamma \vdash A$, i.e. suppose that we can derive the type judgement $\Gamma \vdash u : A$. Give derivation trees corresponding to the following type judgements:

1. $\Gamma \vdash \lambda y.(y \otimes u) : B \multimap (B \otimes A)$
2. $\Gamma \vdash (\lambda x \lambda y.(y \otimes x))\, u : B \multimap (B \otimes A)$

(b) Let $\mathcal{C}$ be a compact closed category (so you can use cups and caps and the closed structure is given as explained in the first page, in particular $X \multimap Y = X^* \otimes Y$). Suppose that the term u is interpreted as an arrow $u : \Gamma \longrightarrow A$ in $\mathcal{C}$. Give the arrows of type $\Gamma \longrightarrow B^* \otimes B \otimes A$ corresponding to each of the lambda terms 1. and 2. above. You can give this using diagrammatic representation. (Also, you are allowed to be a bit cavalier with the use of swaps and the rules related to this).

(c) Note that the second λ-term in question (a), $(\lambda x \lambda y.(y \otimes x))\, u$, can be β-reduced to $(\lambda y.(y \otimes x))[u/x] = \lambda y.(y \otimes u)$, the first λ-term. Show that the two arrows in the category $\mathcal{C}$ calculated in question (b) corresponding to each of the two λ-terms are the equal. You can use the diagrammatic representation and the rules of the diagrammatic calculus to establish this equality.

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References

[1] S. Abramsky and B. Coecke. Categorical quantum mechanics. In Kurt Engesser, Dov Gabbay, and Daniel Lehmann, editors, *Handbook of Quantum Logic and Quantum Structures*, pages 261–324. North-Holland, Elsevier, 2011.

[2] S. Abramsky and N. Tzevelekos. Introduction to categories and categorical logic. In B. Coecke, editor, *New Structures for Physics*, pages 3–94. Springer Lecture Notes on Physics (813), 2011.